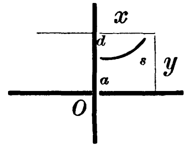


$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right), \qquad (1)$$

we have

$$s = \frac{a}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right), \qquad (2)$$

whence

$$s^2 = y^2 - a^2 = (y-a)(y+a). \qquad (3)$$

Now suppose the ends of the tape to be on a level, and that d, the deflection of the curve, is measured; the length of the tape is $2s$, and s and d are the known quantities. But $y-a=d$, hence from (3), $y+a=s^2\div d$; whence

$$y = \frac{s^2+d^2}{2d}, \qquad \text{and } a = \frac{s^2-d^2}{2d}. \qquad (4)$$

From (1) and (2), $e^{\frac{x}{a}} = \frac{y+s}{a}$,

and substituting from (4),

$$e^{\frac{x}{a}} = \frac{s^2+d^2+2ds}{s^2-d^2} = \frac{s+d}{s-d};$$

whence

$$x = a\log\frac{s+d}{s-d} = \frac{s^2-d^2}{2d}\log\frac{s+d}{s-d}, \qquad (5)$$

or, putting L for $2x$, the length to be measured, and S for $2s$, the whole length of the tape,

$$L = \frac{S^2-4d^2}{4d}\log\frac{S+2d}{S-2d}, \qquad (6)$$

the logarithm being Naperian.

If we develop the logarithm by the formula

$$\log\frac{1+x}{1-x} = 2x\left[1+\frac{x^2}{3}+\frac{x^4}{5}+\ldots\right],$$

we have

$$L = S\left[1-\frac{(2d)^2}{S^2}\right]\left[1+\frac{(2d)^2}{3S^2}+\frac{(2d)^4}{5S^4}+\ldots\right]$$

$$= S\left[1-\frac{2(2d)^2}{3S^2}-\frac{2(2d)^4}{3.5S^4}-\frac{2(2d)^6}{5.7S^6}-\frac{2(2d)^8}{7.9S^8}-\ldots\right]. \qquad (7)$$

The second term is of course the same that occurs in the case of a circular arc.

SOLUTIONS OF PROBLEMS IN NUMBER THREE.

Solutions of problems in No. 3 have been received, as follows:

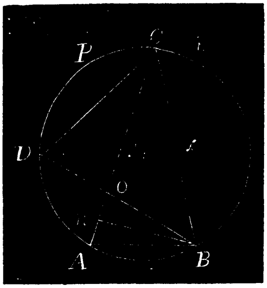
From Prof W. P. Casey, 259, 264; Alexander S. Christie, 261; Geo. Eastwood, 265; W. V. McKnight, 259; W. L. Marcy, 263; E. B. Seitz, 264, 265; Prof. J. Scheffer, 259 and answer to query at page 55. Also, Prof. J. H. Kershner and R. J. Adcock each sent ans. to query at p. 55.

259. "In a triangle, one side, $=400$ ft., and the two adjacent angles, $70°$ and $80°$, are given; to compute the other two sides without the aid of trigonometry."

SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

About the triangle describe a circle, and bisect the angle ABC by the line BD, join DC and draw Bn perpendicular to AC.

Angle $ABC=80°$, $\angle BAC=70°$; $\therefore$ $\angle ACB=30°$; $\therefore$ $BO=AB=a$. Let $BC(=BD)=x$ and $OC(=DC)=y$; then $x:a::y:OA=\frac{ay}{x}$; $\therefore$ $On=\frac{ay}{2x}$, and $a^2-\frac{a^2y^2}{4x^2}=\overline{Bn}^2=\frac{1}{4}x^2$.



$$\therefore 4a^2-\frac{a^2y^2}{x^2}=x^2; \qquad (1)$$

but (Eucl., B. VI.) $AB\times BC=AO\times OC+BO^2$;

$$\therefore y^2=x^2-ax. \qquad (2)$$

By substituting this value for y^2 in (1) we get

$$x^3-a^2x=a^3;$$

from which we find $x=751.752$ft. The other values of x are imaginary.

The value of y $(=CO)=514.227$ft. and $AO=273.615$ft., $\therefore$ $AC=787.842$ft.

Cor. AB is the side of a hexagon inscribed in the circle $ABCD$, and if we allow ph to be its opposite side, then will arc $pC=2Ch$, and the chord pC is the side of a figure of nine sides, inscribed in the circle, and that of Ch the side of a figure of eighteen sides.

$Ah=$diam.$=2AB=800$, $Ch=\sqrt{(Ah^2-Ch^2)}=\sqrt{[(800)^2-(787.842)]}$; and $Cp=\sqrt{(Bp^2-BC^2)}=\sqrt{[(800)^2-(751.752)^2]}$; $\therefore$ an arc of $60°$ may be trisected without the aid of trigonometry.

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260. "A sphere, radius r, rolls down the concave arc of a circle, radius R. At the beginning of the motion, the center of the sphere is on the horizontal diameter of the circle. Find the time of descent of the sphere in terms of the coordinates of its center."

No solution of this problem has been received.

[Putting φ for the arc of the circle over which the sphere has rolled in any given time t, and θ for the angular rotation of the sphere in the same time, we shall have

$$r(\theta-\varphi)=R\varphi. \qquad (1)$$

The equations of motion are (see Routh's Rigid Dynamics, page 103)

$$(R-r)\frac{d^2\varphi}{dt^2}=+g\sin\varphi-\frac{F}{m},\qquad(2)$$

$$(R-r)\left(\frac{d\varphi}{dt}\right)^2=-g\cos\varphi+\frac{R'}{m},\qquad(3)$$

$$\frac{d^2\theta}{dt^2}=\frac{rF}{mk^2}.\qquad(4)$$

Substituting for $\frac{d^2\theta}{dt^2}$ in (4), its equivalent, $\frac{R+r}{r}\cdot\frac{d^2\varphi}{dt^2}$, as found from (1),

we have $\frac{R+r}{r}\cdot\frac{d^2\varphi}{dt^2}=\frac{rF}{mk^2}$, $\therefore\ \frac{d^2\varphi}{dt^2}=\frac{r}{R+r}\cdot\frac{rF}{mk^2}=\frac{r^2F}{(R+r)mk^2}$. (5)

Combining (5) and (2) we get

$$\frac{r^2F}{(R+r)mk^2}=\frac{g\sin\varphi}{R-r}-\frac{F}{m(R-r)};\ \therefore\ F=\frac{mk^2(R+r)g\sin\varphi}{(R-r)r^2+(R+r)k^2}.$$

Substituting for F in (2) its value as here found and reducing, we get

$$\frac{d^2\varphi}{dt^2}=\frac{g\sin\varphi}{R-r}\left(1-\frac{(R+r)k^2}{(R-r)r^2+(R+r)k^2}\right)=\frac{5g}{7R-3r}\sin\varphi.\qquad(6)$$

Multiplying by $2d\varphi\div dt$ and integrating we get, after correcting,

$$\left(\frac{d\varphi}{dt}\right)^2=\frac{10g}{7R-3r}\left(1-\cos\varphi\right)=a(1-\cos\varphi).$$

Therefore we have $dt=\frac{d\varphi}{\sqrt{[a(1-\cos\varphi)]}}$, whence $t=\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{[a(1-\cos\varphi)]}}$,

$$=\frac{1}{\sqrt{a}}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{(1-\cos\varphi)}}=\frac{1}{\sqrt{a}}\left[\frac{1}{\sqrt{2}}\log\frac{\sqrt{(1+\cos\varphi)}-\sqrt{2}}{\sqrt{(1-\cos\varphi)}}\right]_0^{\frac{\pi}{2}},$$

$$=(2\div\sqrt{a})\log(2-\sqrt{2}).\text{—Ed.]}$$

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261. "A curve of the nth degree rolls upon any curve whatever: to determine the degree and position of the locus of the centers of curvature of all the elements described, at any given instant, by the points of the rolling curve."

SOLUTION BY ALEX. S. CHRISTIE, U. S. COAST SURVEY, WASHINGTON, D. C.

In the more general case of a curve of the nth degree,

$$0=u_0+u_1+u_2+\ldots+u_p+\ldots+u_r\qquad(1)$$

(where u_p denotes the aggregate of terms of the pth degree), carried by any curve rolling on any other, if we assume, as x and y axes, the tangent and normal to the point of contact of the two curves, and reckon θ from the common normal, a property of the circle of inflexions gives

$$\rho=\frac{e\cos\theta\,\rho_1}{e\cos\theta+\rho_1}=\frac{ey}{k^2}\rho_1,$$

where ρ is the vector of a point of the carried curve, ρ_1 the vector of the corresponding point of the sought locus, e the diameter of the circle of inflexion, and $k^2 = ey+x^2+y^2$.

Hence the required locus is

$$0 = u_0k^{2n}+u_1eyk^{2(n-1)}+\ldots+u_n(ey)^n, \tag{2}$$

which reduces, when (1) rolls on the fixed curve, to

$$0 = y^2\left\{-\frac{2c_0r}{r-e}k^{2(n-1)}+u_2k^{2(n-2)}+u_3eyk^{2(n-3)}+\ldots+u_n(ey)^{n-2}\right\}, \tag{3}$$

where c_0 is the coefficient of x^2 in (1), and r is the radius of curvature of the fixed curve at the point of contact.

Equation (2) holds in form in whatever manner (1) moves. When (1) is a right line, (2) is a parabola, an hyperbola, or an ellipse, according as (1) touches, cuts, or does not meet the circle of inflexions, and (3) gives the parabola $y^2 = 0$, a special case of the parabola of (2).

When (1) is a conic, (2) is of the 4th degree, and (3) is a conic touching the rolling conic at the instantaneous center of rotation, with the parabola $y^2 = 0$ $(x = dx^2)$.

In general (2) and (3) are each of the $(2n)$th deg. with n and $n+1$ branches, respectively, passing through the origin tangent to the rolling curve.

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262. No solution received.

263. "On the 17th of Aug., 1878, at $5^h\,20^m$ A. M., an observation of polaris was taken in Lat. 37°30′, Long. 107° W. from Greenwich, with an engineer's transit; from the meridian thus obtained, the Sun's azimuth at $6^h23^m53^s$, A. M., was 79°10′30″. What was the error of the watch, and of the meridian?

SOLUTION BY W. L. MARCY, C. E., LEADVILLE, COL.

The right ascension and declination of polaris, for Aug. 17, 1878, Greenwich mean noon, were, respectively, $1^h\,14^m\,35^s$, and 88° 39′ 34″. The diff. of time for 107° of longitude is $7^h\,8^m$; hence the observation of polaris was taken at 20 minutes after Greenwich mean noon. Correcting the R. S. of polaris to correspond with the time of observation, I find the azimuth of polaris for lat. 37° 30′ to be 46′ 57″, nearly.

With the bearing of polaris at the time of observation, corrected for his azimuth, I lay off the approximate meridian, LP'.

Reducing the time of the Sun-observation to Greenwich time, I find it to be $1^h\,31^m\,53^s$ after Greenwich mean noon. The Sun's right ascension and declination for Aug. 17, 1878, were, respectively, 9^h 46′ 48.6″, and 13°23′

56.9″, from which, correcting for the time past noon, the Sun's declination at the time of the observation is found to be 13° 22′ 43.4″. Hence, if S represents the apparent place of the Sun, we have the two sides LP' and SP in the spherical triangle $LP'S$ and the angle SLP', opposite SP' to find angle $SP'L = 90° \, 14' \, 13\frac{1}{3}''$. Converting this angle into time, and correcting for equation of time, I get $6^h \, 2^m \, 55.5^s$ for mean time by first calculation. The difference between this and the assumed time is $20^m \, 57\frac{1}{2}^s$.

Repeating the calculation, with corrected time, I get for azim. of polaris, at second trial, 38′ 23″, at third trial, 38′ 48″, at fourth trial, 38′ 46.4″; and for the time of Sun-observation, I find, from second cal. $6^h \, 3^m \, 55.3^s$, and from third calculation, $6^h \, 3^m \, 52.4^s$. Hence I find the error of the watch to be $20^m \, 1.6^s$, and that of the meridian, 8′ 10.4″.

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264. "It is required to divide a given straight line into three parts such that the triangle formed of them shall have its circumscribing circle a minimum and its inscribed circle a maximum."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

Let $2a$, x, y and $2a—x—y$ = the length of the line and the three pieces, respectively, u = the area of the triangle formed, v and w = the radii of the circumscribed and inscribed circles. Then we have

$$u = \sqrt{[a(a-x)(a-y)(x+y-a)]}, \tag{1}$$

$$u = aw, \tag{2}$$

$$4vu = xy(2a-x-y). \tag{3}$$

Differentiating (1), (2) and (3), regarding y a constant, we find

$$\frac{du}{dx} = \frac{a(a-y)(2a-2x-y)}{u}, \tag{4}$$

$$\frac{dw}{dx} = \frac{1}{a} \cdot \frac{du}{dx} = 0, \tag{5}$$

$$\frac{dv}{dx} = \frac{y(2a-2x-y)}{4u} - \frac{v}{u} \cdot \frac{du}{dx} = 0. \tag{6}$$

From (4) and (5), or from (4) and (6) we find

$$2a - 2x - y = 0. \tag{7}$$

Differentiating (1), (2) and (3), regarding x a constant, we find

$$\frac{du}{dy} = \frac{a(a-x)(2a-x-2y)}{u}, \tag{8}$$

$$\frac{dw}{dy} = \frac{1}{a} \cdot \frac{du}{dy} = 0, \tag{9}$$

$$\frac{dv}{dy}=\frac{x(2a-x-2y)}{4u}-\frac{v}{u}\cdot\frac{du}{dy}=0. \tag{10}$$

From (8) and (9), or from (8) and (10) we find

$$2a-x-2y=0. \tag{11}$$

From (7) and (11) we find $x=\frac{2}{3}a,\ y=\frac{2}{3}a,\ 2a-x-y=\frac{2}{3}a.$

[Prof. E. J. Edmunds writes, in relation to this problem, "It seems to me that one of the conditions is unnecessary. For the second condition alone determines the three divisions of the line, each of which must be equal to one-third of the line.

"It is well known that $r=s\div p$; and r will be a max. when s is a max. If x, y, z are the three parts, $s=\sqrt{[p(p-x)(p-y)(p-z)]}$, where $x+y+z=2p$. S will be a max. with $(p-x)(p-y)(p-z)$, and this will be when $p-x=p-y=p-z$; $\therefore x=y=z=\frac{2}{3}p$."]

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265. "At age a, a person takes out a life policy of k dollars, for which he agrees to pay an annual premium of p dollars. At age $a+n$ he is not able to make his annual payment and wants to sell, or surrender, his life-policy for full-paid insurance policy. If the n premiums that have been paid be each viewed as the sum of a series of infinitely small annuities, payable at infinitely small intervals of time, within each year, what ought to be the present value of the new policy?"

SOLUTION BY GEORGE EASTWOOD, SAXONVILLE, MASS.

Let i = rate of interest allowed by the insurance company, $f(a)$ = number of persons alive at age a, as per mortality tables, ${}^tC_a$ = present worth of a life series of \$1 per annum, payable at equal intervals t, and resting on a life of age a. Then the probability that the insured will be alive at age $a+t$ will be expressed by $f(a+t)\div f(a)$, and ${}^tC_a$ will be defined by

$${}^tC_a=t\left[\frac{f(a+t)}{f(a)}\left(1+i\right)^{-t}+\frac{f(a+2t)}{f(a)}\left(1+i\right)^{-2t}+\ldots\right]. \tag{1}$$

If in (1) we diminish t indefinitely, we shall have

$${}^0C_a=\int_0^\infty\frac{f(a+x)}{f(a)}\left(1+i\right)^{-x}dx. \tag{2}$$

Adapting this to Euler's formula, we deduce an approx. int. of the form

$$\int_0^\infty\frac{f(a+x)}{f(a)}\left(1+i\right)^{-x}dx=\sum_{x=0}^{x=\infty}\frac{f(a+x)}{f(a)}\left(1+i\right)^{-x}+\frac{t}{2}-\frac{t^2}{12}\frac{f'(a)}{f(a)}\cdots \tag{3}$$

By virtue of eq's (1) and (2), the first two right-hand terms of (3) give

$${}^0C_a = {}^tC_a + \tfrac{1}{2}t, \tag{4}$$

and when $t = 0$,
$${}^0C_a = C_a + \tfrac{1}{2}. \tag{5}$$

Whence, from (4) and (5)
$${}^tC_a = C_a + \tfrac{1}{2} - \tfrac{1}{2}t. \tag{6}$$

2. Let $P_a =$ single premium to be paid by the insured to the company to insure payment to him, by the latter, if $1 at his decease. The probability that the insured will be alive at age $a + x$, and that he will *not* be alive at age $a+x+dx$, being, respectively, $f(a + x) \div f(a)$, and $1-[f(a + x + dx) \div f(a + x)]$, the probability that he will decease, during the time dx, will be expressed by

$$\frac{f(a+x)}{f(a)}\left[1-\frac{f(a+x+dx)}{f(a+x)}\right] = -\frac{d\,f(a+x)}{f(a)}. \tag{7}$$

By multiplying eq. (7) by $(1 + i)^{-x}$, we shall have what French Actuaries term the mathematical hope of the event whose probability we are in search of. If now we take the sum of all the analogous hopes, relatively to the values of x comprised between zero and the extreme limit of the mortality tables, we shall have

$$P_a = -\int_{x=0}^{x=\infty} \frac{df(a+x)}{f(a)}\left(1+i\right)^{-x}. \tag{8}$$

Integrating by parts, equation (8) gives

$$P_a = \frac{1}{f(a)}\left[f(a) - \int_0^{\infty} f(a+x)(1+i)^{-x} \log(1+i)dx\right]. \tag{9}$$

The integral included in eq. (9) is the continuous annuity ${}^0C_a$ of eq. (2); hence
$$P_a = 1 - {}^0C_a \log(1+i).$$

Hence, also
$$P_{a+n} = 1 - {}^0C_{a+n} \log(1+i). \tag{10}$$

3. In our problem, p being the annual premium, the value of the unpaid premiums at the time of default of payment will be represented by

$$p(1+C_{a+n});$$

while the value of the whole insurance, considered as a commercial commodity, will be expressed by

$$KP_{a+n}.$$

The insured then may be considered to have a claim upon the company at age $a+n$ worth, at his decease,

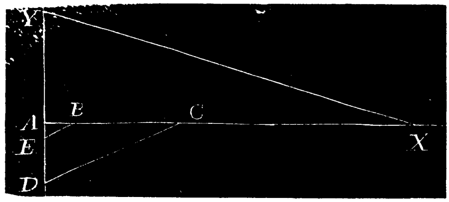
$$KP_{a-n} - p(1+C_{a+n}).$$

If X denote its present value, then obviously,

$$1 : P_{a+n} :: X : KP_{a+n} - p(1+C_{a+n}).$$

Whence
$$X = K - [p \div (P_{a+n})](1+C_{a+n})$$
$$= K - [p(1+C_{a+n}) \div \{ 1 - {}^0C_{a+n} \log(1-i) \}]$$
$$= K - [p(1+C_{a+n}) \div \{ 1 - {}^tC_{a+n} \log(1+i) \} + \tfrac{1}{2}t].$$

NOTE BY PROF. P. E. CHASE.—To construct π, approximately, draw the rectangular axes AX, AY; make $AX = 60$, and take, on the line AX, $AB = 3$, $AC = 20$. On the axis AY, lay off, negatively, $AD = 9$. Join CD and draw BE parallel to CD; make $EY = AC = 20$ and join XY. Then is $XY \div AC = 3.14158499 = \pi$, very nearly.



[The demonstration, received in answer to query at p. 55, is deferred, for want of room, to a future number.]

PROBLEMS

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266. By PROF. J. H. KERSHNER, *Mercersburg, Pa.*—Sum the series,

$$\cos 2\theta \cos \theta + \frac{\cos^2 2\theta \cos 3\theta}{2\,.\,3} + \frac{1\,.\,3}{2\,.\,4} \cos^3 2\theta \,\frac{\cos 5\theta}{5} + \ldots \text{ to infinity.}$$

267. By WM. HOOVER, *Bellefontaine, Ohio.*—What was the duration of a building and loan association in which, for the first eight years, money was loaned at an average premium of \$45 per share (of \$200), interest paid being 6 per cent.

268. By E. B. OPDYCKE, *Pulaski, Ohio.*—A conical vessel whose upper diameter is a, lower diameter b $(a > b)$ and height h, contains a quantity of water. Find the greatest perpendicular depth of the water when the vessel is so tipped on its bottom as to balance, the vessel being uniform in thickness, and of one material.

269. By GEO. H. HARVILL, *Bonner, La.*—Show that the equations

$$x = (r_1 - r_2)\cos\varphi + mr_2 \cos\left(\frac{r_1 - r_2}{r_2}\,\varphi\right),$$

$$y = (r_1 - r_2)\sin\varphi - mr_2 \sin\left(\frac{r_1 - r_2}{r_2}\,\varphi\right),$$

represent the prolate and curtate hypocycloids; and also that when $r_1 = 2r_2$ the curve becomes the ellipse

$$\frac{x^2}{(1+m)^2 r_2^2} + \frac{y}{(1-m)^2 r_2^2} = 1.$$

270. By PROF. E. J. EDMUNDS, *New Orleans, La.*—Find the locus of the centres of circles tangent to a parabola and to the tangent at its vertex.